Noise and Decoherence in Quantum Two-Level Systems

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Motivated by recent experiments with Josephson-junction circuits we reconsider decoherence effects in quantum two-level systems (TLS). On one hand, the experiments demonstrate the importance of $1/f$ noise, on the other hand, by operating at symmetry points one can suppress noise effects in linear order. We, therefore, analyze noise sources with a variety of power spectra, with linear or quadratic coupling, which are longitudinal or transverse relative to the eigenbasis of the unperturbed Hamiltonian. To evaluate the dephasing time for transverse $1/f$ noise second-order contributions have to be taken into account. Manipulations of the quantum state of the TLS define characteristic time scales. We discuss the consequences for relaxation and dephasing processes.

1. Introduction

The dynamics of quantum two-level systems has always been at the focus of interest, but recently has attracted increased attention because of the ideas of quantum computing. Several systems have been suggested as physical realizations of quantum bits allowing for the needed controlled manipulations, and for some of them first elementary steps have been demonstrated in experiments (see reviews in Ref. [1]). A crucial requirement is the preservation of phase coherence in the presence of a noisy environment. In solid-state realizations, incl. those based on superconducting circuits, major noise sources are the electronic control circuit as well as material-specific fluctuations, e.g., in the substrate. In many cases one lacks a detailed microscopic description of the noise source, but frequently it is sufficient to model the environment by a bath of harmonic oscillators with frequency spectrum adjusted to reproduce the observed power spectrum. The resulting "spin-boson models" have been studied in the literature (see the reviews [2,3]), in particular the one with linear coupling and "Ohmic" spectrum. On the other hand, several recent experiments appear to be described by spin-boson models with more general couplings and different power spectra. In this article we, therefore, describe these extensions of the spin-boson model and analyze how more general noise sources and coupling schemes affect relaxation and dephasing processes.

In the following Section we summarize what is known about relaxation and dephasing processes for spin-boson models with linear coupling. We add several extensions,

including a discussion how these processes are induced by specific quantum manipulations. We also present some exact results, shedding light on the important question of low-temperature dephasing. This analysis, as well as the rest of this article are applicable to any quantum two-state system subject to noise with appropriate coupling and power spectrum. However, since the extensions have been motivated by recent experiments with Josephson-junction qubits, we shortly present in Section 3 the physics of these systems. We indicate how, by proper choice of the operating point, one can tune to zero the linear coupling to the environment. Section 4 covers the extensions of the spin-boson model, which were motivated by the recent experiments. This includes noise sources with a variety of power spectra, with linear or quadratic coupling, which are longitudinal or transverse relative to the eigenbasis of the unperturbed Hamiltonian. For transverse $1/f$ noise we evaluate also second-order contributions, which turn out to be comparable to the lowest order. We conclude with a summary. The technical parts of our work, which are based on an analysis of the time evolution of the reduced density matrix of the quantum two-state system will be described in a separate publication.

2. Spin-boson model

In this section we review the theory and properties of the spin-boson model, which have been studied extensively before (see the reviews [2,3]). A quantum two-level system coupled to an environment is modeled by a spin degree of freedom in a magnetic field coupled linearly to an oscillator bath with Hamiltonian

$$\mathcal{H} = \mathcal{H}_{\mathrm{ctrl}} + \sigma_z \sum_j c_j (a_j + a_j^\dagger) + \mathcal{H}_{\mathrm{b}} \,. \tag{1}$$

The controlled part is

$$\mathcal{H}_{\mathrm{ctrl}} = -\frac{1}{2} B_z \, \sigma_z - \frac{1}{2} B_x \, \sigma_x = -\frac{1}{2} \Delta E \left(\cos\theta \, \sigma_z + \sin\theta \, \sigma_x \right) , \tag{2}$$

while the oscillator bath is described by

$$\mathcal{H}_{\mathrm{b}} = \sum_j \hbar\omega_j \, a_j^\dagger a_j \,. \tag{3}$$

In the second form of Eq. (2) we introduced the mixing angle $\theta \equiv \tan^{-1}(B_x/B_z)$, which depends on the direction of the magnetic field in the x-z-plane, and the energy splitting between the eigenstates, $\Delta E = \sqrt{B_x^2 + B_z^2}$. In the standard spin-boson model it is assumed that the bath "force" operator $X = \sum_j c_j (a_j + a_j^\dagger)$ couples linearly to σ_z.

In thermal equilibrium the Fourier transform of the symmetrized correlation function of this operator is given by

$$S_X(\omega) \equiv \langle [X(t), X(t')]_+ \rangle_\omega = 2\hbar J(\omega) \coth \frac{\hbar\,\omega}{2 k_{\mathrm{B}} T} \,. \tag{4}$$

Here the bath spectral density has been introduced, defined by

$$J(\omega) \equiv \frac{\pi}{\hbar} \sum_j c_j^2 \, \delta(\omega - \omega_j) \,. \tag{5}$$

At low frequencies it typically has a power-law behavior. Of particular interest is the "Ohmic dissipation", corresponding to a spectrum which is linear at low frequencies up to some high-frequency cutoff ω_c,

$$J(\omega) = \frac{\pi}{2}\,\alpha\,\hbar\omega\,\Theta(\omega_c - \omega)\ . \tag{6}$$

The dimensionless parameter α reflects the strength of dissipation. In a physical system it depends on the amplitude of the noise and the coupling strength. Here we concentrate on weak damping, $\alpha \ll 1$, since this limit is relevant for quantum-state engineering. Still we distinguish two regimes: the Hamiltonian-dominated regime, which is realized when $\Delta E \gg \alpha\, k_{\rm B}T$, and the noise-dominated regime, which is realized, e.g., at degeneracy points where $\Delta E \to 0$.

In the Hamiltonian-dominated regime, $\Delta E \gg \alpha\, k_{\rm B}T$, it is natural to describe the evolution of the system in the eigenbasis, $|0\rangle$ and $|1\rangle$, of $\mathcal{H}_{\rm ctrl}$:

$$\begin{aligned} |0\rangle &= \quad\cos\frac{\theta}{2}\,|\!\uparrow\rangle + \sin\frac{\theta}{2}\,|\!\downarrow\rangle \\ |1\rangle &= -\sin\frac{\theta}{2}\,|\!\uparrow\rangle + \cos\frac{\theta}{2}\,|\!\downarrow\rangle\ . \end{aligned} \tag{7}$$

Denoting by τ_x and τ_z the Pauli matrices in the eigenbasis, we have

$$\mathcal{H} = -\frac{1}{2}\Delta E\,\tau_z + (\sin\theta\ \tau_x + \cos\theta\ \tau_z)\ X + \mathcal{H}_{\rm b}\ . \tag{8}$$

2.1. Relaxation and dephasing

Two different time scales describe the evolution in the spin-boson model [2–5]. The first is the dephasing time scale τ_φ. It characterizes the decay of the off-diagonal elements of the qubit's reduced density matrix $\hat\rho(t)$ in the preferred eigenbasis (7), or, equivalently of the expectation values of the operators $\tau_\pm \equiv (1/2)(\tau_x \pm i\tau_y)$. Dephasing processes lead to the following long-time dependence:

$$\langle\tau_\pm(t)\rangle \equiv {\rm tr}\,[\tau_\pm\hat\rho(t)] \propto \langle\tau_\pm(0)\rangle\ e^{\mp i\Delta E t/\hbar}\ e^{-t/\tau_\varphi}\ . \tag{9}$$

(Other time dependences will be discussed below). The second, the relaxation time scale $\tau_{\rm relax}$, characterizes how the diagonal entries tend to their thermal equilibrium values:

$$\langle\tau_z(t)\rangle - \tau_z(\infty) \propto e^{-t/\tau_{\rm relax}}\ , \tag{10}$$

where $\tau_z(\infty) = \tanh(\Delta E/2k_{\rm B}T)$.

In Refs. [2,3] the dephasing and relaxation times for the spin boson model with Ohmic spectrum were evaluated in a path-integral technique. In the regime $\alpha \ll 1$ it is easier to employ the perturbative (diagrammatic) technique developed in Ref. [6] and the standard Bloch-Redfield approximation. The rates are [1]

$$\Gamma_{\rm relax} \equiv \tau_{\rm relax}^{-1} \quad = \pi\alpha\ \sin^2\theta\ \frac{\Delta E}{\hbar}\coth\frac{\Delta E}{2k_{\rm B}T} \quad = \frac{1}{\hbar^2}\sin^2\theta\ S_X\left(\omega = \Delta E/\hbar\right)\ , \tag{11}$$

$$\Gamma_\varphi \equiv \tau_\varphi^{-1} \quad = \tfrac{1}{2}\,\Gamma_{\rm relax} + \pi\alpha\ \cos^2\theta\ \frac{2k_{\rm B}T}{\hbar} \quad = \frac{1}{2}\,\Gamma_{\rm relax} + \frac{1}{\hbar^2}\cos^2\theta\, S_X(\omega = 0)\ . \tag{12}$$

[1] Note that in the literature usually the evolution of $\langle\sigma_z(t)\rangle$ has been studied. To establish the connection to the results (11,12) one has to substitute Eqs. (9,10) into the identity $\sigma_z = \cos\theta\ \tau_z + \sin\theta\ \tau_x$. Here we neglect renormalization effects, since they are weak for $\alpha \ll 1$.

We observe that only the "transverse" τ_x-component of the fluctuating field, proportional to $\sin\theta$, induces transitions between the eigenstates (7) of the unperturbed system. Thus the relaxation rate (11) is proportional to $\sin^2\theta$. The "longitudinal" τ_z-component of the fluctuating field, proportional to $\cos\theta$, does not induce relaxation processes. It does contribute, however, to dephasing since it leads to fluctuations of the eigenenergies and, thus, to a random relative phase between the two eigenstates. This is the origin of the contribution to the "pure" dephasing rate Γ^*_φ (12), which is proportional to $\cos^2\theta$. Relaxation and pure dephasing contribute to the rate $\Gamma_\varphi = \frac{1}{2}\Gamma_{\rm relax} + \Gamma^*_\varphi$.

The last forms of Eqs. (11) and (12) express the two rates in terms of the noise power spectrum at the relevant frequencies. These are the level spacing of the two-state system and zero frequency, respectively. The expressions apply in the weak-coupling limit for spectra which are regular at these frequencies (see Section 4 for $1/f$ noise). In general, fluctuations with frequencies in an interval of width $\Gamma_{\rm relax}$ around $\Delta E/\hbar$ and of width Γ_φ around zero are involved.

The equilibration is due to two processes, excitation $|0\rangle \to |1\rangle$ and relaxation $|1\rangle \to |0\rangle$, with rates $\Gamma_{+/-} \propto \langle X(t)X(t')\rangle_{\omega=\pm\Delta E/\hbar}$. Both rates are related by a detailed balance condition, and the equilibrium value $\tau_z(\infty)$ depends on both. On the other hand, $\Gamma_{\rm relax}$ is determined by the sum of the two rates, i.e., the symmetrized noise power spectrum S_X.

In the environment-dominated regime, $\Delta E \ll \alpha\, k_{\rm B}T$, the coupling to the bath is the dominant part of the total Hamiltonian. Therefore, it is more convenient to discuss the problem in the eigenbasis of the observable σ_z to which the bath is coupled. The spin can tunnel incoherently between the two eigenstates of σ_z. One can again employ the perturbative analysis [6] but use directly the Markov instead of the Bloch-Redfield approximation. The resulting rates are given by

$$\begin{aligned} \Gamma_{\rm relax} &\approx B_x^2/(2\pi\hbar\alpha k_{\rm B}T)\ , \\ \Gamma_\varphi &\approx 2\pi\alpha k_{\rm B}T/\hbar\ . \end{aligned} \tag{13}$$

In this regime the dephasing is much faster than the relaxation. In fact, we observe that as a function of temperature the dephasing and relaxation rates evolve in opposite directions. The α-dependence of the relaxation rate is an indication of the Zeno (watchdog) effect [7]: the environment frequently "observes" the state of the spin, thus preventing it from tunneling.

2.2. Sensitivity to the initial state

To get further insight into dephasing phenomena we analyze some experimental scenarios. In particular, we discuss their sensitivity to details of the preparation of the initial state (cf., e.g., Ref. [8]). Such a preparation is a necessary ingredient in an experiment probing dephasing processes directly. We observe that depending on the time scales of the preparation part of the bath oscillators (the fast ones) may follow the two-level system adiabatically, merely renormalizing its parameters, while others (the slow ones) lead to dephasing. We illustrate these results by considering the exactly solvable limit $\theta = 0$.

Dephasing processes are contained in the time evolution of the quantity $\langle\tau_+(t)\rangle$ obtained after tracing out the bath. This quantity can be evaluated analytically for $\theta = 0$ (in which case $\tau_+ = \sigma_+$), for an initial state described by a factorized density matrix $\hat\rho(t=0) =$

$\hat{\rho}_{\rm spin} \otimes \hat{\rho}_{\rm bath}$ (i.e., the TLS and the bath are disentangled). We find

$$\langle \sigma_+(t) \rangle \equiv \mathcal{P}(t)\, e^{-i\Delta E t}\, \langle \sigma_+(0) \rangle \,, \qquad \text{where} \qquad \mathcal{P}(t) = {\rm Tr}(e^{i\Phi(0)/2} e^{-i\Phi(t)} e^{i\Phi(0)/2} \hat{\rho}_{\rm bath})\,. \tag{14}$$

Here the bath operator Φ is defined as

$$\Phi \equiv i \sum_j \frac{2c_j}{\hbar\omega_j} (a_j^\dagger - a_j)\,, \tag{15}$$

and its time evolution is determined by the bare bath Hamiltonian, $\Phi(t) = e^{i\mathcal{H}_{\rm b}t} \Phi e^{-i\mathcal{H}_{\rm b}t}$. To derive Eq. (14) we made use of a unitary transformation by $U \equiv \exp\left(-i\sigma_z \Phi/2\right)$. In the new basis the bath and the TLS decouple, which allows for the exact solution.

The expression (14) applies for any state of the bath (as long as it is factorized from the spin). In particular, we can assume that the spin was initially (for $t \le 0$) kept in the state $|\!\uparrow\rangle$ and the bath has relaxed to the thermal equilibrium distribution for this spin value: $\hat{\rho}_{\rm bath} = \hat{\rho}_\uparrow \equiv Z_\uparrow^{-1} e^{-\beta \mathcal{H}_\uparrow}$, where $\mathcal{H}_\uparrow = \mathcal{H}_{\rm b} + \sum_j c_j (a_j + a_j^\dagger)$. In this case we can rewrite the density matrix as $\hat{\rho}_{\rm bath} = e^{i\Phi/2} \hat{\rho}_{\rm b} e^{-i\Phi/2}$, with the density matrix of the decoupled bath given by $\hat{\rho}_{\rm b} \equiv Z_{\rm b}^{-1} e^{-\beta\mathcal{H}_{\rm b}}$, and obtain

$$\mathcal{P}(t) = P(t) \equiv {\rm Tr}\left(e^{-i\Phi(t)}\, e^{i\Phi}\, \hat{\rho}_{\rm b}\right)\,. \tag{16}$$

The latter expression (with Fourier transform $P(E)$) has been studied extensively in the literature [2,9–12]. It can be expressed as $P(t) = \exp K(t)$, where

$$K(t) = \frac{4}{\pi\hbar} \int_0^\infty d\omega\, \frac{J(\omega)}{\omega^2} \left[\coth\left(\frac{\hbar\omega}{2k_B T}\right) (\cos\omega t - 1) - i \sin\omega t \right]\,. \tag{17}$$

For an Ohmic bath (6) at nonzero temperature and $t > \hbar/2k_{\rm B}T$, it reduces to

$$\text{Re}K(t) \approx -\frac{S_X(\omega=0)}{\hbar^2} t = -\pi\,\alpha\, \frac{2k_{\rm B}T}{\hbar}\, t\,. \tag{18}$$

Thus we reproduce Eq. (9) with the dephasing rate Γ_φ given by (12) in the limit $\theta = 0$.

This result is easy to understand at the Golden rule level. Pure dephasing processes are associated with transitions during which only the oscillators change their state, while the spin remains unchanged. Since no energy is exchanged only oscillators with frequencies near zero contribute. An analysis of the Golden rule shows that the range is the inverse of the relevant time scale, in the present case given by Γ_φ. Since for an Ohmic model the spectral density $J(\omega)$ vanishes linearly as $\omega \to 0$, exponential dephasing is found only at nonzero temperature $T > 0$. The situation is different in the presence of sub-Ohmic noise where, even at $T = 0$, the low-frequency oscillators may cause exponential dephasing (see the discussion in Sec. 4.1).

For vanishing bath temperature, $T = 0$, one still finds a decay of $\langle\sigma_+(t)\rangle$, governed by Re $K(t) \approx -2\,\alpha\, \ln(\omega_c t)$, which implies a power-law decay

$$\langle \sigma_+(t) \rangle = (\omega_c t)^{-2\alpha} e^{-i\Delta E t/\hbar} \langle \sigma_+(0) \rangle \text{ for } t > 1/\omega_c\,. \tag{19}$$

This result is beyond the Golden rule. In fact all oscillators up to the high-frequency cutoff ω_c contribute. While the power-law decay is a weak effect, it highlights the influence of

the initial state preparation and coherent manipulations on the dephasing. This will be particularly important in the sub-Ohmic and the super-Ohmic regime, as we will describe later.

So far we discussed various initial states without specifying how they are prepared. We now consider some possibilities. A (theoretical) one is to keep the spin and bath decoupled until $t = 0$. In equilibrium the initial state which then enters Eq. (14) is the product state $\hat{\rho}_{\mathrm{bath}} = \hat{\rho}_{\mathrm{b}} \propto e^{-\beta\mathcal{H}_{\mathrm{b}}}$. A more realistic possibility is to keep the bath coupled to the spin, while forcing the latter, e.g., by a strong external field, to be in a fixed state, say $|\!\uparrow\rangle$. Then, at $t = 0$, a sudden pulse of the external field is applied to change the spin state to a superposition, e.g., $\frac{1}{\sqrt{2}}(|\!\uparrow\rangle + |\!\downarrow\rangle)$. If the bath has no time to respond the resulting state at $T = 0$ is $|\mathrm{i}\rangle = \frac{1}{\sqrt{2}}(|\!\uparrow\rangle + |\!\downarrow\rangle) \otimes |\mathrm{g}_\uparrow\rangle$, where $|\mathrm{g}_\uparrow\rangle$ is the ground state of $\mathcal{H}_\uparrow$. Both components of this initial wave function now evolve in time according to the Hamiltonian (1). The first, $|\!\uparrow\rangle \otimes |\mathrm{g}_\uparrow\rangle$, which is an eigenstate of (1), acquires only a trivial phase factor. The time evolution of the second component is more involved. Up to a phase factor it is given by $|\!\downarrow\rangle \otimes \exp(-i\mathcal{H}_\downarrow t/\hbar)\,|\mathrm{g}_\uparrow\rangle$ where $\mathcal{H}_\downarrow \equiv \mathcal{H}_{\mathrm{b}} - \sum_j c_j(a_j + a_j^\dagger)$. As the state $|\mathrm{g}_\uparrow\rangle$ is not an eigenstate of $\mathcal{H}_\downarrow$, entanglement between the spin and the bath develops, and the coherence between the states of the spin is reduced by the factor $|\,\langle \mathrm{g}_\uparrow| \exp(-i\mathcal{H}_\downarrow t/\hbar)\,|\mathrm{g}_\uparrow\rangle\,| < 1$.

In a real experiment of the type discussed the preparation pulse takes a finite time, during which the oscillators partially adjust to the changing spin state. For instance, the $(\pi/2)_x$-pulse which transforms the state $|\!\uparrow\rangle \to \frac{1}{\sqrt{2}}(|\!\uparrow\rangle + |\!\downarrow\rangle)$, can be accomplished by putting $B_z = 0$ and $B_x = \Delta$ for a time span $\pi\hbar/2\Delta$. In this case the oscillators with (high) frequencies, $\hbar\omega_j \gg \Delta$, follow the spin adiabatically. In contrast, the oscillators with low frequency, $\hbar\omega_j \ll \Delta$, do not change their state. Assuming that the oscillators can be split into these two groups, we see that just after the $(\pi/2)_x$-pulse the state of the system is $\frac{1}{\sqrt{2}}\left(|\!\uparrow\rangle \otimes \left|\mathrm{g}_\uparrow^{\mathrm{h}}\right\rangle + |\!\downarrow\rangle \otimes \left|\mathrm{g}_\downarrow^{\mathrm{h}}\right\rangle\right) \otimes \left|\mathrm{g}_\uparrow^{\mathrm{l}}\right\rangle$ where the superscripts 'h' and 'l' refer to the high- and low-frequency oscillators, respectively. Thus, we arrive at a factorized initial state of the type as discussed above. However, only the low-frequency oscillators are factorized from the spin and contribute to dephasing during the subsequent evolution, while the high-frequency modes give rise to renormalization effects. Thus we reproduce the decay of $\langle\sigma_+(t)\rangle$ described above (i.e., for an Ohmic bath a power-law decay), however, with a preparation-dependent cutoff frequency $\hbar\omega_c = \Delta$. We may add that similar considerations apply for the analysis of the measurement process.

2.3. Response functions

In the limit $\theta = 0$ we can also calculate exactly the linear response of $\tau_x = \sigma_x$ to a weak magnetic field in the x-direction, $B_x(t)$:

$$\chi_{xx}(t) = \frac{i}{\hbar}\,\Theta(t)\langle\tau_x(t)\tau_x(0) - \tau_x(0)\tau_x(t)\rangle\ . \tag{20}$$

Using the equilibrium density matrix

$$\hat{\rho}^{\mathrm{eq}} = \frac{1}{1+e^{-\beta\Delta E}}\left[|\!\uparrow\rangle\langle\uparrow\!| \otimes \hat{\rho}_\uparrow + e^{-\beta\Delta E}\,|\!\downarrow\rangle\langle\downarrow\!| \otimes \hat{\rho}_\downarrow\right]\ , \tag{21}$$

where $\hat{\rho}_\uparrow \propto \exp(-\beta\mathcal{H}_\uparrow)$ is the bath density matrix adjusted to the spin state $|\uparrow\rangle$ and similar for $\hat{\rho}_\downarrow$, we obtain the susceptibility

$$\chi_{xx}(t) = -\frac{2\hbar^{-1}\Theta(t)}{1+e^{-\beta\Delta E}}\,\mathrm{Im}\left[P(t)e^{-i\Delta Et} + e^{-\beta\Delta E}P(t)e^{i\Delta Et}\right] \ . \tag{22}$$

The imaginary part of its Fourier transform is

$$\chi''_{xx}(\omega) = \frac{1}{2(1+e^{-\beta\Delta E})}\left[P(\hbar\omega - \Delta E) + e^{-\beta\Delta E}P(\hbar\omega + \Delta E)\right] - ...(-\omega) \ . \tag{23}$$

At $T = 0$ and positive values of ω we use the expression for $P(E)$ from Ref. [12] to obtain

$$\chi''_{xx}(\omega) = \frac{1}{2}P(\hbar\omega - \Delta E) = \Theta(\hbar\omega - \Delta E)\frac{e^{-2\gamma\alpha}(\hbar\omega_c)^{-2\alpha}}{2\Gamma(2\alpha)}(\hbar\omega - \Delta E)^{2\alpha-1} \ . \tag{24}$$

We observe that the dissipative part χ''_{xx} has a gap ΔE, which corresponds to the minimal energy needed to flip the spin, and a power-law behavior as ω approaches the threshold. This behavior of $\chi''_{xx}(\omega)$ parallels the *orthogonality catastrophe* scenario [13]. It implies that the ground state of the oscillator bath for different spin states, $|\mathrm{g}_\uparrow\rangle$ and $|\mathrm{g}_\downarrow\rangle$, are *macroscopically* orthogonal. In particular, for an Ohmic bath (6) we recover the behavior typical for the problem of the X-ray absorption in metals [13].

The case of perpendicular noise, $\theta = \pi/2$, has been treated in the literature (see e.g. Refs. [3,14]). Then, in the coherent (Hamiltonian-dominated) regime, $\chi''_{xx}(\omega)$ shows a Lorentzian peak around the frequency corresponding to the (renormalized) level splitting ΔE. The width of the peak is determined by the dephasing associated with the relaxation processes, i.e., by $\Gamma_\varphi = \Gamma_{\mathrm{relax}}/2$ as given in Eqs. (11) and (12).

3. Josephson junction qubits

In this section we describe, by way of a specific example, some principles of quantum-state manipulations and the influence of various noise sources. For this purpose we consider a Josephson-junction qubit based on a superconducting single-charge box displayed in Fig. 1a (for an extended discussion see Ref. [15]). It consists of a small superconducting island coupled via Josephson junctions with effective coupling energy $E_{\mathrm{J}}(\Phi_{\mathrm{x}})$ and the junction capacitance C_{J}) to a superconducting lead, and via a gate capacitor C_{g} to a gate voltage source V_{g} [16]. The Hamiltonian of this system is

$$\mathcal{H} = \frac{(Q-Q_{\mathrm{g}})^2}{2C} - E_{\mathrm{J}}(\Phi_{\mathrm{x}})\cos\varphi \quad \text{with} \quad Q = \frac{\hbar}{i}\frac{\partial}{\partial(\hbar\varphi/2e)}. \tag{25}$$

The excess Cooper-pair charge $Q = 2ne$ on the island, which is an integer multiple of $2e$, and the phase difference φ across the Josephson junction are conjugate variables [17], as indicated by the second part of Eq. (25). The charging energy, with a characteristic scale $E_C \equiv e^2/2C$, depends on the total capacitance of the island, $C = C_{\mathrm{J}} + C_{\mathrm{g}}$, and is controlled by the gate charge $Q_{\mathrm{g}} = C_{\mathrm{g}}V_{\mathrm{g}}$. We allow for a circuit with several junctions such that the Josephson coupling energy can be controlled by an applied flux Φ_{x}.

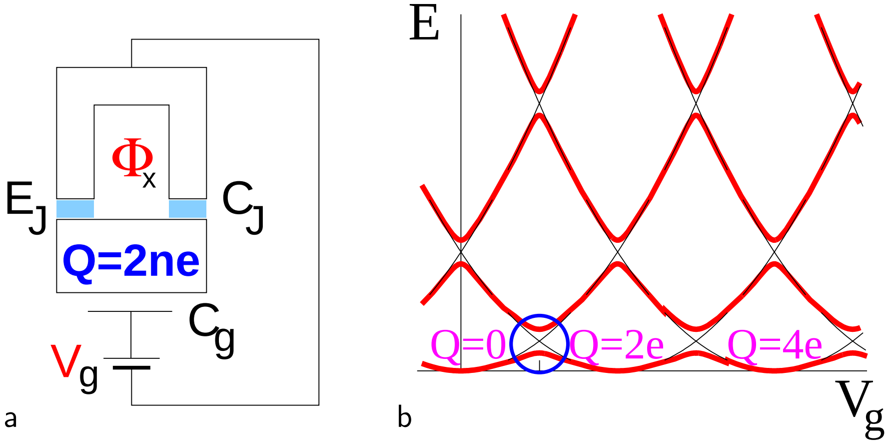

Figure 1. a) Josephson charge qubit; b) eigenenergies as functions of the gate voltage (for fixed Φ_x)

When the charging energy dominates, $E_C \gg E_\mathrm{J}$, it is convenient to rewrite Eq. (25) in the eigenbasis of the number n of Cooper-pair charges [18]

$$\mathcal{H} = \sum_{n=-\infty}^{\infty} \frac{(2ne - Q_\mathrm{g})^2}{2C} |n\rangle\langle n| - \frac{1}{2} E_\mathrm{J}(\Phi_\mathrm{x}) \Big(|n\rangle\langle n+1| + |n+1\rangle\langle n| \Big). \tag{26}$$

In this limit it is obvious that eigenenergies form approximately parabolic bands as a function of the gate charge Q_g, which can be labeled by n and are shifted relatively to each other by $2e$ (see Fig. 1b). Near the degeneracy points, where two parabolas cross, the bands are split by an amount equal to $E_\mathrm{J}(\Phi_\mathrm{x})$. (A similar picture had been proposed in a different context, that of a current-biased low-capacitance Josephson junction in Refs. [19,20].) If we concentrate on the vicinity of such a degeneracy point the system reduces to a quantum two-state system with Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \Delta E_\mathrm{ch}(V_\mathrm{g})\, \sigma_z - \frac{1}{2} E_\mathrm{J}(\Phi_\mathrm{x})\, \sigma_x \,. \tag{27}$$

We note that by controlling the gate voltage we vary the difference in the charging energy of the two adjacent charge states, $\Delta E_\mathrm{ch}(V_\mathrm{g})$, and hence what corresponds to an effective magnetic field in the z-direction. Similarly the applied flux controls the Josephson coupling energy, corresponding to a control of the effective magnetic field in the x-direction. In several recent experiments [21–23] quantum manipulations of Josephson charge qubits described by (27) have been performed.

Also for general relations between the charging energy and the Josephson coupling energy we have a well-defined quantum-mechanical problem. The system has a sequence of levels $E_i(V_\mathrm{g}, \Phi_\mathrm{x})$, which depend periodically on the gate voltage and the applied flux and can be controlled in this way. At low temperatures and for suitable applied ac-fields it may be sufficient to concentrate on the lowest two of these states [24]. The other extreme, $E_\mathrm{C} \ll E_\mathrm{J}$, is realized in Josephson flux qubits, studied by other groups [25,26]. Here the control has to be achieved solely by applied fluxes (possibly, different fluxes applied to different parts of the circuit).

The control of the qubit by applied voltages and fluxes is always accompanied by noise. As is clear from Eq. (27) a fluctuating control voltage $\delta V(t)$ couples to σ_z, while the flux noise $\delta\Phi(t)$ couples to σ_x. Since these noises are derived from linear circuits they are Gaussian and can be modeled by a bath of harmonic oscillators. They are completely characterized by their power spectra, which in turn depend on the impedance $Z(\omega)$ and the temperature of the relevant circuits.

For instance, the gate voltage fluctuations in the control circuit satisfy

$$S_V(\omega) = 2\,\mathrm{Re}\{Z_\mathrm{t}(\omega)\}\,\hbar\omega\,\coth\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right) \,. \tag{28}$$

They are governed by the total impedance of the circuit as seen from the qubit, $Z_\mathrm{t}^{-1}(\omega) \equiv i\omega C_\mathrm{qb} + Z^{-1}(\omega)$, which is determined by the external circuit $Z(\omega)$ and the capacitance $C_\mathrm{qb} \equiv (C_\mathrm{J}^{-1} + C_\mathrm{g}^{-1})^{-1}$ of the qubit in the circuit. At the degeneracy point the voltage fluctuations couple linearly to the qubit, via $\mathcal{H}_1 = -2e(C_\mathrm{g}/C_\mathrm{qb})\,\delta V\,\sigma_z$. In general, this noise is neither longitudinal nor transverse in the eigenbasis of $\mathcal{H}_\mathrm{ctrl}$, rather it is characterized by the angle $\theta_V = \tan^{-1}(E_\mathrm{J}/\Delta E_\mathrm{ch})$. At low frequencies the circuit usually behaves as a resistor $Z(\omega) = R_V$, and the results for Ohmic dissipation discussed in the previous section apply. They are characterized by the dimensionless parameter

$$\alpha_V = \frac{4R_V}{R_\mathrm{K}}\left(\frac{C_\mathrm{g}}{C_\mathrm{qb}}\right)^2 \,, \tag{29}$$

which is determined both by the strength of voltage fluctuations of the environment ($\propto R_V$) and by the coupling of these fluctuations to the qubit ($\propto C_\mathrm{g}/C_\mathrm{qb}$). We note that the circuit resistance is compared to the (typically much higher) quantum resistance $R_\mathrm{K} = h/e^2 \approx 25.8$ kΩ. At frequencies considered here the typical impedance of the voltage circuit is $R_V \approx 50\ \Omega$. Furthermore, the effect of fluctuations is weakened if the gate capacitance is chosen small, $C_\mathrm{g} \ll C_\mathrm{J}$. Taking the ratio $C_\mathrm{g}/C_\mathrm{J} = 10^{-2}$ one reaches a dissipation as weak as $\alpha_V \approx 10^{-6}$, allowing, theoretically, for $\sim 10^6$ coherent single-bit manipulations.

Similarly the fluctuations of the external flux can be expressed by the effective impedance of the current circuit which supplies the flux and the mutual inductance characterizing the coupling. Choosing the impedance purely real, R_I, we have

$$\alpha_\Phi = \frac{R_\mathrm{K}}{4R_I}\left(\frac{M}{\Phi_0}\frac{\partial E_\mathrm{J}(\Phi_x)}{\partial\Phi_x}\right)^2 \,, \tag{30}$$

and $\theta_\Phi = \tan^{-1}(\Delta E_\mathrm{ch}/E_\mathrm{J})$. For an estimate we take $R_I \sim 100\ \Omega$ of the order of the vacuum value. For $M \approx 0.01 - 0.1$ nH and $E_\mathrm{J}^0/k_\mathrm{B} \approx 0.1$ K we obtain $\alpha_\Phi \approx 10^{-6} - 10^{-8}$.

In spite of these favorable estimates for the dephasing rates the actual experiments suffer from further, stronger noise sources. For instance, in charge qubits [27] an important contribution comes from "background charge fluctuations". Their origin may be charge transfer processes between impurities in the substrate, which typically lead to a $1/f$ power spectrum. This noise source can be modeled by further 2-state quantum systems [28,29], however, when many of them couple weakly to the TLS they can be approximated again by an oscillator bath with appropriate spectrum. We will analyze this model in the next section.

There are ways to reduce the effect of the fluctuating environment [24], which can be demonstrated by considering the 2-state Hamiltonian (27): The idea is to tune the gate voltage to the degeneracy point $V_{\rm g0}$, where the difference in the charging energy vanishes, $\Delta E_{\rm ch}(V_{\rm g0}) = 0$, and, furthermore, to tune the applied flux to a point where the effective Josephson coupling energy has an extremum, $\partial E_{\rm J}(\Phi_{\rm x})/\partial \Phi_{\rm x} = 0$. Hence the Hamiltonian is

$$\mathcal{H} = -\frac{1}{2} E_{\rm J}(\Phi_{\rm x0})\, \sigma_x - \frac{1}{2} \frac{\partial \Delta E_{\rm ch}(V_{\rm g})}{\partial V_{\rm g}}\, \delta V\, \sigma_z - \frac{1}{4} \frac{\partial^2 E_{\rm J}(\Phi_{\rm x})}{\partial \Phi_{\rm x}^2}\, \delta\Phi^2\, \sigma_x\ . \tag{31}$$

At this operating point the energy difference between the two relevant states depends only quadratically on the fluctuations in either the gate voltage $\delta V(t)$ or the applied flux $\delta\Phi(t)$. On the other hand, one still can perform quantum-state manipulations by applying small-amplitude ac gate voltages, $V_{\rm ac}(t)$, as is usual in NMR. (Only the noise at frequency $\Delta E/\hbar$ couples through this channel; its effect is much weaker than that of the pure dephasing if $1/f$ noise is dominant.) By employing this idea the group in Saclay [24] has recently observed remarkably long dephasing times of the order of 1 μsec. This progress has been one of our motivations to study noise effects in more general situations.

Another motivation to extend the analysis of the noise comes from the investigation of quantum measurement devices. Except when a measurement is being performed, the detector should be decoupled from the TLS as much as possible. Since it is hard to achieve complete decoupling, one usually turns to zero the linear coupling, but higher-order terms may still be present and affect the quantum dynamics. As an example we consider a single-electron transistor (SET), which can serve as a detector of the quantum state of a Josephson charge qubit [15,30]. The interaction between qubit and SET is given by $\mathcal{H}_{\rm int} = E_{\rm int}\sigma_z N$, where $N = \sum_i a^\dagger_{\rm I,i} a_{\rm I,i}$ is the number of electrons on the central island of the SET, and $E_{\rm int}$ denotes the capacitive coupling energy. The electron number N changes due to tunneling to the leads of the SET. Thus, in general the SET involves fermionic baths, but in some cases the problem can be mapped onto a spin-boson model: Consider the off-state of the SET with no transport voltage. In this case tunneling processes are suppressed at temperatures below the Coulomb gap of order E_C by the Coulomb blockade, and, classically, the charge of the central island is fixed at, say, $N = 0$. On the other hand, virtual higher-order processes (cotunneling) involve other charge states, e.g. $N = 1$, which makes N noisy. This situation can be mapped onto (the two lowest states of) a harmonic oscillator, with eigenfrequency equal to the Coulomb gap E_C, coupled to a dissipative bath. Since the interaction between the qubit and the oscillator depends on the occupation number operator of this oscillator, it corresponds to a nonlinear coupling in the formulation presented above.

In this context we can also mention the recent proposal [31] of a SQUID-based device which allows switching off the linear coupling of a detector to a flux qubit, leaving also in this example only nonlinear coupling.

These and further examples provide the motivation to study noise effects in more general situations than those covered by the usual spin-boson model. In the next Section we, therefore, investigate Ohmic circuit noise as well as $1/f$ noise, which couple in a linear or nonlinear way to the two-state quantum system. The noise can be longitudinal or transverse, which means in the eigenbasis of the TLS they couple to τ_z or τ_x, respectively.

4. Extensions of the spin-boson model

The spin-boson model has been studied mostly for the specific case where a bath with Ohmic spectrum is coupled linearly to the spin degree of freedom. One reason is that linear damping, proportional to the velocity (Ohmic), is encountered frequently in real systems. Another is that suitable systems with Ohmic damping show a quantum phase transition at a critical strength of the dissipation, with $\alpha_{\rm cr} \sim 1$. On the other hand, in the context of quantum-state engineering we should concentrate on systems with weak damping, but allow for general coupling and general spectra of the fluctuations. In these cases we find qualitatively novel results. For instance, a two-level system with sub-Ohmic damping (e.g., $1/f$ noise) still shows coherent oscillations, in spite of a general belief that it should localize in one of its states. Vise versa, in the super-Ohmic regime, where usually dephasing effects are believed negligible, we still find that manipulations with sharp pulses do influence dephasing. Finally we consider nonlinear coupling of the noise source to the qubit, which is important at symmetry points [24]. Here, for definiteness we will consider only one source of fluctuations at a time.

4.1. Sub-Ohmic and $1/f$ noise

As mentioned above several experiments with Josephson circuits revealed in the low-frequency range the presence of $1/f$ noise. While the origin of this noise may be different in different circuits and requires further analysis, it appears that in several cases it derives from "background charge fluctuations". In this case the noise of the gate charge may be quantified as $S_{Q_{\rm g}}(\omega) = \alpha_{1/f} e^2/\omega$. Recent experiments [27] yield $\alpha_{1/f} \sim 10^{-7}$–$10^{-6}$ and indicate that the $1/f$ frequency dependence may extend up to high values, of the order of the level spacing of the TLS.

One way to model the noise is to use an oscillator bath, where in the case of nonequilibrium $1/f$ noise the bath temperature $T_{\rm b}$ should be treated as an adjustable parameter. For instance, a bath with $J(\omega) = (\pi/2)\alpha\hbar\omega_s$ gives at $\omega \ll k_{\rm B}T_{\rm b}/\hbar$ the $1/f$ noise spectrum

$$S_X(\omega) = \frac{E_{1/f}^2}{|\omega|} \,. \tag{32}$$

with $E_{1/f}^2 = 2\pi\alpha\hbar\omega_s k_{\rm B}T_{\rm b}$.

Such a model is a particular example of the so-called *sub-Ohmic* damping, defined by spectral densities

$$J_s(\omega) = (\pi/2)\,\hbar\alpha\,\omega_s^{1-s}\omega^s \qquad \text{with } s < 1\,. \tag{33}$$

Sub-Ohmic damping was considered earlier, for $0 < s < 1$, in the framework of the spin-boson model [2,3], but did not attract much attention. It was argued that in the presence of sub-Ohmic dissipation coherence would be totally suppressed, transitions between the states of the two-level system would be incoherent and take place only at finite temperatures. At zero temperature the system should be localized in one of the eigenstates of σ_z. The localization results from the fact that the bath renormalizes the off-diagonal part of the Hamiltonian B_x to zero. However, this scenario, while correct for intermediate to strong damping, is not correct for weak damping. Indeed the NIBA approximation [2], which was designed to cover intermediate to strong damping, fails in the weak-coupling

limit for transverse noise. In contrast a more sophisticated renormalization procedure [32] predicts coherent behavior. In the context of quantum-state engineering we are interested in precisely this *coherent sub-Ohmic* regime. We will demonstrate that the simple criterion, which was used in Section 2 to distinguish between regimes of coherent and incoherent dynamics (i.e., the comparison of Γ_φ^* and ΔE), can be used also for sub-Ohmic environments.

We consider the transient coherent dynamics of a spin coupled to a sub-Ohmic bath. If all the oscillators of the bath follow the spin adiabatically, the renormalized matrix element $B_x \propto \exp[-\int d\omega J(\omega)/\omega^2]$ is indeed suppressed, implying an incoherent dynamics. For a sub-Ohmic bath the integral diverges at low frequencies. However, a finite preparation time $\hbar/\Delta$ (cf. the discussion in Sec. 2) provides a low-frequency cutoff Δ for the oscillators that contribute to the renormalization, thus leading to a finite B_x. The low-frequency oscillators, $\hbar\omega_j \ll \Delta$, contribute to the dephasing. Their effect is only weakly sensitive to the cutoff Δ, since the relevant integrals are dominated by very low frequencies.

For longitudinal noise ($\theta = 0$) with a $1/f$ spectrum one obtains for the function $\mathcal{P}(t)$ [Eq. (14)], with logarithmic accuracy [33], $\mathrm{Re}\ln\mathcal{P}(t) \approx -(E_{1/f}t)^2|\ln(\omega_{\mathrm{ir}}t)|/\pi\hbar^2$, where ω_{ir} is the (intrinsic) infrared cutoff frequency for the $1/f$ noise.[2] From this decay law one can deduce a dephasing rate,

$$\Gamma_\varphi^* \approx \frac{1}{\hbar}\, E_{1/f}\sqrt{\frac{1}{\pi}\ln\frac{E_{1/f}}{\omega_{\mathrm{ir}}}}\,. \tag{34}$$

Next we consider transverse noise, $\theta = \pi/2$. As in the Ohmic regime we compare the energy splitting $\Delta E = B_x$ to the pure dephasing rate Γ_φ^* (34) in order to determine whether the Hamiltonian- or noise-dominated regime is realized. In the former case, $\Delta E \gg \hbar\Gamma_\varphi^*$, coherent oscillations are expected. This is confirmed by a perturbative analysis, similar to the one used to derive Eqs. (11,12), which gives in first order in α

$$\begin{aligned}\Gamma_{\mathrm{relax}} &= \frac{E_{1/f}^2}{\hbar\,\Delta E}\,,\\ \Gamma_\varphi &= \frac{1}{2}\,\Gamma_{\mathrm{relax}}\,.\end{aligned} \tag{35}$$

A more detailed analysis reveals that in the present problem higher-order contributions to the dissipative rates may be important as well. The reason is that, for $\theta = \pi/2$, the first-order relaxation rate is sensitive to the noise $S_X(\omega)$ only at a (high) frequency $\Delta E/\hbar$ where the $1/f$ noise is weak, whereas in second order the (diverging) low-frequency part of the spectrum becomes important. Employing the diagrammatic technique of Ref. [6], we link the time evolution of the density matrix to a self-energy, Σ. We find that the Laplace transform of the second-order contribution $\Sigma^{(2)}(s)$ diverges as $s \to i\Delta E/\hbar$. Using first- and second-order terms in Σ we solve for the poles of $\tau_+(s) \propto [s - i\Delta E/\hbar - \Sigma(s)]^{-1}$ to describe the short-time behavior and find, with logarithmic accuracy

$$\Gamma_\varphi = a\,\frac{E_{1/f}^2}{\hbar\,\Delta E} \qquad \text{with } a \approx \frac{1}{2\pi}\ln\frac{E_{1/f}^2}{\hbar\,\omega_{\mathrm{ir}}\,\Delta E}\,. \tag{36}$$

[2] In an experiment with averaging over repetitive measurements it may be determined by the time interval t_{av} over which the averaging is performed [24]. In a spin-echo experiment the echo frequency may determine this frequency [27].

Thus, we see that the second-order correction to the self-energy changes the result for the dephasing rate considerably. As a consequence, for transverse noise the ratio $\Gamma_\varphi/\Gamma_{\rm relax}$, which in first order takes the value $1/2$, is shifted substantially towards higher values. It is interesting that in the recent experiments of Ref. [24] a ratio $\Gamma_\varphi/\Gamma_{\rm relax} \approx 3$ was reported. A more detailed description of the second order calculation will be presented elsewhere. Finally we note that Eq. (36) indeed confirms the assumption of underdamped coherent oscillations, as $\hbar\Gamma_{\rm relax}, \hbar\Gamma_\varphi \ll \Delta E$ in the Hamiltonian-dominated regime.

Notice that for a sub-Ohmic bath with $-1 < s < 1$ due to a high density of low-frequency oscillators the dephasing persists at $T_{\rm b} = 0$: $|\langle\tau_+(t)\rangle| \propto \exp[-\alpha(\omega_s t)^{1-s}/(s+1)]$ from which we read off the dephasing rate $\Gamma^*_\varphi \propto [\alpha/(s+1)]^{1/(1-s)}\omega_s$ (cf. Ref. [34]).

In the noise-dominated regime, $\Delta E \ll \hbar\Gamma^*_\varphi$, the dynamics is incoherent as we know from earlier work [2,3]. Thus, our criterion for the coherent regime, $\Delta E \gg \hbar\Gamma^*_\varphi$, is valid also for $1/f$ and sub-Ohmic environments. For $0 < s < 1$, the critical damping strength at which $\Delta E \sim \hbar\Gamma^*_\varphi$ coincides with that obtained in Ref. [32] and with the boundary of the applicability range of NIBA [2].

4.2. Super-Ohmic case

Next we discuss a super-Ohmic bath with, e.g., $s = 2$ and $J_s(\omega) = (\pi/2)\,\hbar\alpha\,\omega^2/\omega_s$. At zero temperature this leads to

$$\mathrm{Re}\ K(t) = \frac{2\alpha}{\omega_s}\left(\frac{\sin(\omega_c t)}{t} - \omega_c\right) . \tag{37}$$

Thus, within a short time of order ω_c^{-1} the exponent Re $K(t)$ saturates at a finite value Re $K(\infty) = -2\alpha\omega_c/\omega_s$. While usually this reduction of the off-diagonal elements of the density matrix is not denoted as dephasing (since coherent oscillations persist), from the point of view of quantum-state engineering it is a relevant loss of phase coherence, which introduces errors. We also note that this reduction strongly depends on the frequency cutoff ω_c. Recalling that this cutoff is determined by the quantum-state manipulation (e.g., the width of a $\pi/2$-pulse) we observe that in the super-Ohmic regime the (weak) dephasing is strongly sensitive to the details of the preparation procedure.

4.3. Nonlinear coupling to the noise source

As motivated in the preceding section we study next a generalization of the spin-boson model to include nonlinear coupling of the noise source to the variables of the TLS. The Hamiltonian in the eigenbasis of the unperturbed system (with Pauli matrices denoted by τ) thus reads

$$\mathcal{H} = -\frac{1}{2}\Delta E\tau_z + (\sin\theta\ \tau_x + \cos\theta\ \tau_z)\ \frac{X^2}{E_0} + \mathcal{H}_{\rm b}\ . \tag{38}$$

Since the bath "force" operator $X = \sum_j c_j(a_j + a_j^\dagger)$ has dimensions of energy, for nonlinear coupling a new energy scale, E_0, appears. In specific systems this energy scale is a characteristic scale of the system. E.g., for a Josephson charge qubit coupled quadratically to the flux noise [Eq. (31)] we have $E_0 = E_{\rm J}$.

When considering a quadratic coupling we need to know the statistical properties of $X^2(t)$. While X is Gaussian-distributed, X^2 is not. However, for a first-order perturbative

analysis this fact is irrelevant and we can assume that X^2 is Gaussian as well, with a width that is fixed by the correlator $S_{X^2}(\omega) \equiv \langle\{X^2(t), X^2(t')\}\rangle_\omega$. For baths with spectral densities which are regular at low frequencies it can be shown that this approximation is sufficient to describe the initial stage of the dephasing process for $\Gamma_\varphi t \ll 1$, and it provides a reliable estimate for the dephasing time. For singular spectra (e.g., for $1/f$ noise) a more detailed analysis is needed, and non-Gaussian corrections may appear. Note that in order to evaluate the symmetrized correlator $S_{X^2}(\omega)$ in general one needs to know the full correlator $\langle X(t)X(t')\rangle$ and not only the symmetrized one $S_X(\omega)$. For an Ohmic spectrum (6) the symmetrized and antisymmetrized spectra are related by detailed balance, and we find

$$S_{X^2}(\omega) \sim \alpha^2 \hbar \max\{(\hbar\omega)^3, (k_{\mathrm{B}}T)^3\}\,. \tag{39}$$

For $1/f$ noise (32) (assuming that the antisymmetric part diverges weaker than $1/\omega$ and can be neglected) we find

$$S_{X^2}(\omega) = \frac{2}{\pi}\frac{E_{1/f}^4}{\omega}\ln\frac{\omega}{\omega_{\mathrm{ir}}}\,. \tag{40}$$

Consider first the longitudinal coupling ($\theta = 0$). For an Ohmic bath at low frequencies we have $S_{X^2}(\omega = 0) \sim \alpha^2\hbar(k_{\mathrm{B}}T)^3$. Therefore, the pure dephasing rate is given by

$$\Gamma_\varphi^* = \frac{S_{X^2}(\omega = 0)}{\hbar^2 E_0^2} \sim \frac{\alpha^2 (k_{\mathrm{B}}T)^3}{\hbar E_0^2}\,. \tag{41}$$

For $1/f$ noise, the correlator $S_{X^2}(\omega)$ also exhibits a $1/\omega$ divergence. Hence in analogy to the linear-coupling case the dephasing is governed by $\mathrm{Re}\ln\mathcal{P}(t) = -[E_{1/f}^2 t\ln(\omega_{\mathrm{ir}}t)/\pi\hbar E_0]^2$, i.e., the characteristic dephasing rate is

$$\Gamma_\varphi^* = \frac{E_{1/f}^2}{\pi\,\hbar\,E_0}\,\ln\frac{E_{1/f}^2}{\omega_{\mathrm{ir}}E_0}\,. \tag{42}$$

In the transverse coupling case ($\theta = \pi/2$) the relaxation and dephasing rates are again given by the noise power spectrum S_{X^2} at frequency $\Delta E/\hbar$. In the lowest order we have $\Gamma_{\mathrm{relax}} = S_{X^2}(\Delta E/\hbar)/E_0^2$ for both Ohmic and $1/f$ noise sources. In the case of $1/f$ noise we should again take into account second-order corrections to the self-energy since they probe the divergent low-frequency part of the noise power spectrum. In this way we get $\Gamma_\varphi = b\,\Gamma_{\mathrm{relax}}$ with $b \sim 1$.

4.4. Summary of results

In the table we summarize our results for the relaxation and pure dephasing rates, Γ_{relax} and Γ_φ^*, of a two-state quantum system subject to the different sources of noise, with coupling which is longitudinal or transverse to the qubit (as compared to its eigenbasis), linear or quadratic, with an Ohmic (6), (39) or a $1/f$ spectrum (32), (40). For the sake of brevity we put $\hbar = k_{\mathrm{B}} = 1$ in the table and omit factors of order one in some expressions. Complete results can be found in the text.

	longitudinal ($\parallel$) (⇒ pure dephasing)	**transverse** ($\perp$) (⇒ relaxation + dephasing)

linear coupling:

	$\mathcal{H} = \frac{1}{2}\Delta E\ \rho_z + X\ \rho_z$	$\mathcal{H} = \frac{1}{2}\Delta E\ \rho_z + X\ \rho_x$
"Ohmic"	$\Gamma^*_\varphi = S_X(\omega = 0) \sim \alpha T$	$\Gamma_{\rm relax} = S_X(\Delta E)$ $\Gamma_\varphi = \Gamma_{\rm relax}/2$
1/f	$\langle \tau_+(t)\rangle \sim e^{-E^2_{1/f}\, t^2\ \ln t}$ $\Gamma^*_\varphi = E_{1/f}\ \ln^{1/2}(E_{1/f}/\omega_{\rm ir})$	$\Gamma_{\rm relax} = S_X(\Delta E)$ $\Gamma_\varphi = a\ \Gamma_{\rm relax}\ \ , a \sim 1$

quadratic coupling:

	$\mathcal{H} = \frac{1}{2}\Delta E\ \rho_z + \frac{1}{E_0} X^2\ \rho_z$	$\mathcal{H} = \frac{1}{2}\Delta E\ \rho_z + \frac{1}{E_0} X^2\ \rho_x$
"Ohmic"	$\Gamma^*_\varphi = \frac{1}{E_0^2} S_{X^2}(\omega = 0) \sim \alpha^2 \frac{T^3}{E_0^2}$	$\Gamma_{\rm relax} = \frac{1}{E_0^2} S_{X^2}(\Delta E)$ $\Gamma_\varphi = \Gamma_{\rm relax}/2$
1/f	$\langle \tau_+(t)\rangle \sim e^{-(E^4_{1/f}/E_0^2)\ t^2\ \ln^2 t}$ $\Gamma^*_\varphi = \frac{E^2_{1/f}}{E_0}\ \ln(E^2_{1/f}/E_0\,\omega_{\rm ir})$	$\Gamma_{\rm relax} = \frac{1}{E_0^2}\ S_{X^2}(\Delta E)$ $\Gamma_\varphi = b\ \Gamma_{\rm relax}\ \ , b \sim 1$

5. Summary

To summarize, motivated by recent experiments on Josephson-jucntion circuits, we have considered dephasing effects in a two-level system due to noise sources with various spectra (incl. Ohmic and $1/f$), coupled in a linear or nonlinear, longitudinal or transverse way to the TLS.

We have shown that the dephasing can be sensitive to the details of the initial-state preparation. For instance, a finite preparation time $\sim \hbar/\Delta$ introduces an upper cutoff Δ on the frequency of environmental modes that contribute to dephasing, whereas the higher-frequency modes merely renormalize the parameters of the TLS. In particular, for an Ohmic environment we find a power-law dephasing at $T = 0$, sensitive to this cutoff frequency.

We have also linked the mentioned renormalization effects to the behavior of a response funtion of the TLS, which exhibits features known for the orthogonality catastrophe, including a power-law divergence above a threshold.

Noise with a $1/f$ spectrum we modeled by a sub-Ohmic bath and found a simple criterion for coherent behavior in sub-Ohmic environments. For transverse $1/f$ noise we found that, due to infrared divergences, one needs to take into account second-order contributions in a perturbative analysis. This can increase substantially the ratio between

the relaxation and dephasing times, which takes the value 1/2 in first-order calculations. These findings may contribute to the understanding of the experimental results of Ref. [24], where a higher ratio (~ 3) was found.

We have also analyzed the dynamics in the case of an environment with Ohmic or $1/f$ noise power, which is coupled nonlinearly to the TLS. In the Ohmic case (e.g., for a single-electron transistor coupled to a Cooper-pair box) the dephasing rate scales as T^3, and by biasing the SET at a special point it can be further suppressed to a T^5-dependence. This should be contrasted, e.g., with a quantum point contact which, even in the off-state, couples linearly to the TLS and influences it strongly, $\Gamma_\varphi \propto T$. We reduced the description of a TLS coupled nonlinearly to a noise source with $1/f$ spectrum to that of a TLS with linear coupling to an effective $1/f$ noise source. The results for the relaxation and dephasing times in this situation are relevant for recent experiments [24].

6. Acknowledgements

We thank M.H. Devoret, D. Esteve, Y. Gefen, D. Golubev, Y. Imry, S.K. Kehrein, A.D. Mirlin, Y. Nakamura, A. Rosch, D. Vion, and A.D. Zaikin for stimulating discussions. The work is part of the EU IST Project SQUBIT and of the **CFN** (Center for Functional Nanostructures) which is supported by the DFG (German Science Foundation). Y.M. was supported by the Humboldt Foundation, the BMBF and the ZIP programme of the German government.

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